# New exact solutions of Einstein–Maxwell equations for magnetostatic fields*

Nisha Goyal† and R. K. Gupta

*School of Mathematics and Computer Applications, Thapar University, Patiala-147004, Punjab, India*

(Received 28 November 2011; revised manuscript received 17 February 2012)

The symmetry reduction method based on the Fréchet derivative of differential operators is applied to investigate symmetries of the Einstein–Maxwell field equations for magnetostatic fields, which is a coupled system of nonlinear partial differential equations of the second order. The technique yields invariant transformations that reduce the given system of partial differential equations to a system of nonlinear ordinary differential equations. Some of the reduced systems are further studied to obtain the exact solutions.

**Keywords:** Einstein–Maxwell equations, symmetry reduction method, exact solutions

**PACS:** 04.40.Nr, 02.20.Sv, 04.20.Jb **DOI:** 10.1088/1674-1056/21/9/090401

## 1. Introduction

Magnetic fields play an important role in the study of astrophysical objects, such as neutron stars, white dwarfs, pulsars, black holes, and galaxies. In fact, several observations show that there are various scenarios in which the magnetic fields and the general relativity can not be neglected. One of them is the presence of strong magnetic fields in active galactic nuclei.[1,2] These nuclei are known to produce more radiation than the rest of the entire galaxy and directly affect its structure and evolution. Another scenario is the production of relativistic collimated jets in the inner regions of accretion discs, which can be explained by considering the magneto-centrifugal mechanisms.[3,4] Also, magnetic fields are important for understanding the interplay between magnetic and thermal processes in strong magnetic neutron stars.[5,6] At least 10% of all neutron stars are born as magnetars, with magnetic fields above $1014G$.[7,8] Analytical models that describe these astrophysical objects are often associated with the solutions of Einstein's equations.[9,10] In the search for more realistic models for the compact stellar systems, the energy–momentum tensor, the source of Einstein's equations, is modified by introducing more complex terms that take into account additional physical properties, for example, electromagnetic fields.

These fields for deriving solutions were first considered by Bonnor[11] for obtaining solutions corresponding to radial and longitudinal fields. Misra and RadhaKrishna[12] have obtained some solutions in which the metric function depends on one independent variable. García and Gonjález[13] have presented a detailed study of the counter-rotating model for generic electrostatic (magnetostatic) axially symmetric thin disks without radial pressure. In this paper, we derive new exact analytic solutions of Einstein–Maxwell equations for purely magnetostatic fields in general relativity.

The simplest metric to describe a static axially symmetric spacetime is Weyl's line element[14]

$$\mathrm{d}s^2 = \mathrm{e}^{-2\phi}[r^2\mathrm{d}\phi^2 + \mathrm{e}^{2\Lambda}(\mathrm{d}r^2 + \mathrm{d}z^2)] - \mathrm{e}^{2\phi}\mathrm{d}t^2, \quad (1)$$

where $\phi$ and $\Lambda$ are functions of $r$ and $z$ only. The Einstein–Maxwell field equations in the geometrized units of $8\pi G = c = \mu_0 = \epsilon_0 = 0$ are given by

$$R_{ab} = T_{ab}, \quad (2)$$

$$T_{ab} = F_{ac}F_b^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}, \quad (3)$$

$$F^{ab}_{,b} = 0, \quad (4)$$

$$F_{ab} = A_{a,b} - A_{b,a}, \quad (5)$$

where $R_{ab}$ is the Ricci tensor, $T_{ab}$ is the electromagnetic energy tensor, $F_{ab}$ is the electromagnetic field tensor that satisfies Maxwell's equations for empty spacetime, and $A_a = (\phi, A, 0, 0)$ is the 4-potential.

*Project supported by the Human Resource Development Group Council of Scientific Industrial Research (CSIR), India (Grant No. 09/677(0014)2009-EMR-1).

†Corresponding author. E-mail: goyal.n104@gmail.com

A comma followed by a lower suffix denotes a partial differentiation with respect to the corresponding variable.

The Einstein–Maxwell equations in the presence of purely magnetostatic fields are equivalent to the system

$$\phi_{rr}+\frac{1}{r}\phi_r+\phi_{zz}-\frac{\mathrm{e}^{2\phi}}{2r^2}(A_r^2+A_z^2)=0, \quad (6)$$

$$A_{rr}-\frac{1}{r}+A_{zz}+2(A_r\phi_r+A_z\phi_z)=0, \quad (7)$$

$$\Lambda_r=r(\phi_r^2-\phi_z^2)+\frac{\mathrm{e}^{2\phi}}{2r}(A_r^2-A_z^2), \quad (8)$$

$$\Lambda_z=2r\phi_r\phi_z+\frac{1}{r}\,\mathrm{e}^{2\phi}A_rA_z, \quad (9)$$

where $A$ is the magnetostatic potential and is a function of $r$ and $z$ only. For solving the nonlinear system (6)–(9), first we solve Eqs. (6) and (7) to obtain $\phi(r,z)$ and $A(r,z)$. After that, by substituting $\phi$ and $A$ into Eqs. (8) and (9), we can obtain $\Lambda(r,z)$.

There are various methods for obtaining the solutions of nonlinear partial deferential equations (PDEs). Some of the important contributions for finding exact solutions of nonlinear PDEs are presented in Refs. [15]–[17]. The effective methods for finding symmetries and constructing exact solutions include the Lie classical approach,[18–22] the symmetry reduction approach,[23,24] the nonclassical approach,[25] and the isovector field method.[26,27] In this paper, we will use the symmetry reduction approach to solve the nonlinear Eqs. (6) and (7).

This paper is organized as follows. In Section 2, the symmetries of Eqs. (6) and (7) in the generalized form are derived, which are then used to obtain the associated Lie algebra of vector fields. In Section 3, we deal with the determination of the transformation group for the reduction of the system of nonlinear PDEs to the system of ordinarily deferential equations (ODEs). Exact solutions are also obtained in that section. In Section 4, we give the discussion and the concluding remarks.

## 2. Symmetry reduction method and optimal system

The symmetry group for a single PDE or a system of PDEs is a continuous group of transformations that act on the space of independent and dependent variables and leave the system invariant. The exact solutions for a system of PDEs are found by solving the reduced system of differential equations involving fewer number of independent variables. The symmetry method has been used to examine the exact solutions for various nonlinear PDEs (refer to Refs. [28] and [29]). Herein, we briefly outline Steinberg's method.[24] For this, we set

$$S_1(\phi)\equiv P(\bar{\boldsymbol{X}},\bar{\boldsymbol{\eta}})\phi_r+Q(\bar{\boldsymbol{X}},\bar{\boldsymbol{\eta}})\phi_z+R(\bar{\boldsymbol{X}},\bar{\boldsymbol{\eta}}), \quad (10)$$

$$S_2(A)\equiv P(\bar{\boldsymbol{X}},\bar{\boldsymbol{\eta}})A_r+Q(\bar{\boldsymbol{X}},\bar{\boldsymbol{\eta}})A_z+S(\bar{\boldsymbol{X}},\bar{\boldsymbol{\eta}}), \quad (11)$$

with $\bar{\boldsymbol{X}}=(r,z)$ and $\bar{\boldsymbol{\eta}}=(\phi,A)$. The Fréchet derivative $\bar{\boldsymbol{F}}=(F_1,F_2)$ of $\bar{\boldsymbol{N}}(\bar{\eta})\equiv(N_1,N_2)$ in the direction of $\bar{\boldsymbol{S}}=(S_1,S_2)$ is given by

$$\bar{\boldsymbol{F}}[\bar{\boldsymbol{N}},\bar{\boldsymbol{\eta}},\bar{\boldsymbol{S}}]=\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bar{\boldsymbol{F}}[\bar{\boldsymbol{N}}(\bar{\boldsymbol{\eta}}+\epsilon\bar{\boldsymbol{S}})]\mid_{\epsilon=0}, \quad (12)$$

which gives

$$\begin{aligned}&F_1(\bar{N}_1,\bar{\boldsymbol{\eta}},\bar{\boldsymbol{S}})\\ &=[S_1]_{rr}+\frac{1}{r}[S_1]_r+[S_1]_{zz}-\frac{\mathrm{e}^{2[S_1]}}{2r^2}\\ &\quad\times\big(A_r[S_2]_z+A_z[S_2]_z+2[S_1](A_r^2+A_z^2)\big), \quad (13)\\ &F_2(\bar{N}_2,\bar{\boldsymbol{\eta}},\bar{\boldsymbol{S}})\\ &=[S_1]_{rr}-\frac{1}{r}[S_1]_r+[S_1]_{zz}+2(A_r[S_1]_r\\ &\quad+\phi_r[S_2]_r+A_z[S_1]_z+\phi_z[S_1]_z). \quad (14)\end{aligned}$$

Substituting $S_1$ and $S_2$ from Eqs. (10) and (11) into Eqs. (13) and (14), when expanded, results in the polynomial expressions in various partial derivatives of $\phi$ and $A$ with respect to the spatial variable. The calculations to obtain these derivatives are tedious, so we list here the simplified set of determining equations for the group infinitesimals $P$, $Q$, $R$, and $S$. From Eq. (13), after equating the coefficients of various derivative terms to zero, we obtain

$$\begin{aligned}&Q_\phi=0,\quad Q_A=0,\quad Q_r=0,\quad Q_{zz}=0,\\ &R_z=0,\quad R_\phi=0,\quad R_A=0,\quad R_r=0,\\ &S_z=0,\quad S_\phi=0,\quad S_A=Q_z-R,\\ &S_r=0,\quad P=rQ_z. \quad (15)\end{aligned}$$

From Eq. (14), keeping in view the consequences on the infinitesimals effected by the above set of equations, we obtain the following set of determining equations:

$$\begin{aligned}&P_z=-Q_r,\quad P_\phi=0,\quad P_A=0,\\ &P_r=Q_z,\quad Q_{zz}=-Q_{rr},\quad R_z=-\frac{Q_r}{2r},\\ &R_A=-\frac{S_{AA}}{2},\quad R_r=\frac{-P+rQ_z}{2r^2}. \quad (16)\end{aligned}$$

Now solving determining Eqs. (15) and (16) for infinitesimals $P$, $Q$, $R$, and $S$, we arrive at the following form of the generalized symmetries:

$$P = ar, \quad Q = az + b,$$
$$R = a - l, \quad S = lA + m, \tag{17}$$

where $a$, $b$, $l$, and $m$ are arbitrary constants.

The Lie algebra associated with Eqs. (6) and (7) consists of the following four vector fields:

$$V_1 = \frac{\partial}{\partial z}, \quad V_2 = \frac{\partial}{\partial A}, \quad V_3 = -\frac{\partial}{\partial \phi} + A\frac{\partial}{\partial A},$$
$$V_4 = r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z} + \frac{\partial}{\partial \phi}. \tag{18}$$

Using these vector fields, we can reduce Eqs. (6) and (7) into a system of ODEs after getting the similarity variable and the similarity solution by solving the characteristic equation

$$\frac{\mathrm{d}r}{P} = \frac{\mathrm{d}z}{Q} = \frac{\mathrm{d}\phi}{-R} = \frac{\mathrm{d}A}{-S}. \tag{19}$$

In general, there are infinite number of subalgebras of this Lie algebra formed from any linear combination of generators $V_j$ $(j = 1, 2, 3, 4)$. And for each subalgebra, we can obtain the reduction by using the characteristic equation (19). However, two algebras are similar if they are connected to each other by a transformation from the symmetry group, in this case, their corresponding invariant solutions are connected to each other by the same transformation. Therefore, it is sufficient to put all similar subalgebras into one class. The set of all these representatives is called an optimal system.[29] We will work out first the optimal system and then embark upon the various reductions associated with the generators in the optimal system. We begin by considering a general element $V = a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4$ of the symmetry algebra and subject it to various adjoint transformations to simplify it as much as possible.[29] The adjoint action is given by the Lie series

$$\mathrm{Ad}(\exp(\epsilon V_i))V_j = V_j - \epsilon[V_i, V_j] + \epsilon^2[V_i, [V_i, V_j]] - \cdots, \tag{20}$$

where $[V_i, V_j] = V_iV_j - V_jV_i$ is the commutator for the Lie algebra, and $\epsilon$ is a parameter.

The optimal system consists of the following eight basic vector fields:

$$\text{(i)}V_1, \quad \text{(ii)}V_2, \quad \text{(iii)}V_2 + V_1, \quad \text{(iv)}V_2 - V_1,$$
$$\text{(v)}V_3, \quad \text{(vi)}V_3 + V_1, \quad \text{(vii)}V_3 - V_1,$$
$$\text{(viii)}V_4 + \alpha V_3, \tag{21}$$

where $\alpha$ is an arbitrary constant. Because of the discrete symmetry $(r, z, \phi, A) \to (-r, z, \phi, A)$, vector fields (iv) and (vii) are similar to vector fields (ii) and (vi), respectively, in the optimal system, we confine ourselves to the remaining six essential vector fields while neglecting the other two.

# 3. Exact solutions of Einstein–Maxwell equations for magnetostatic fields

The similarity variables and the similarity solutions can be obtained by solving the characteristic equation (19). The general solution of the equation involves three constants; one becomes the new independent variable $\zeta$, and the other two are $F$ and $G$, which are dependent variables.

In this section, we will present the similarity variables and the similarity solutions for all eight essential vector fields of the optimal system (21). The reductions of Eqs. (6) and (7) into systems of ODEs are obtained for each vector field in the optimal system. Some exact solutions for these ODEs are also given in this section.

## 3.1. Vector field $V_1$

Corresponding to this vector field, the forms of the similarity variable and the similarity solution are

$$\zeta = r, \ \phi(r, z) = F(\zeta), \ A(r, z) = G(\zeta).$$

By using the above similarity variable and similarity solution in Eqs. (6) and (7), the reduced system of ODEs is obtained as follows:

$$F''(\zeta) + \frac{F'(\zeta)}{r} - \frac{\mathrm{e}^{2F(\zeta)}G'^2(\zeta)}{2r^2} = 0, \tag{22}$$
$$G''(\zeta) - \frac{G'}{\zeta} + 2F'(\zeta)G'(\zeta) = 0. \tag{23}$$

The solution for the above reduced system of ODEs is

$$F(\zeta) = \frac{1}{2}\log\zeta - \frac{1}{2}\log\left(\frac{\mathrm{e}^{-2c_1}}{2c_2^2\cosh\left(\dfrac{\log\zeta - c_3}{c_2}\right)^2\zeta}\right) + c_1, \tag{24}$$

$$G(\zeta) = \frac{\sinh\left(\dfrac{\log\zeta - c_3}{2c_2}\right)}{\mathrm{e}^{2c_1}c_2\cosh\left(\dfrac{\log\zeta - c_3}{2c_2}\right)} + c_4. \tag{25}$$

Hence, the solution of Eqs. (6) and (7) is given by

$$\phi(r,z) = \frac{1}{2}\log r - \frac{1}{2}\log\left(\frac{\mathrm{e}^{-2c_1}}{2c_2^2\cosh\left(\dfrac{\log r - c_3}{c_2}\right)^2 r}\right) + c_1, \quad (26)$$

$$A(r,z) = \frac{\sinh\left(\dfrac{\log r - c_3}{2c_2}\right)}{\mathrm{e}^{2c_1}c_2\cosh\left(\dfrac{\log r - c_3}{2c_2}\right)} + c_4, \quad (27)$$

$$\Lambda(r,z) = \log r - c_3 + 2\log\left(\cosh\left(\frac{\log r - c_3}{2c_2}\right)\right) + \frac{\log r}{4c_2^2} - \frac{c_3}{4c_2^2}, \quad (28)$$

where $c_1$, $c_2$, $c_3$, and $c_4$ are arbitrary constants.

### 3.2. Vector field $V_2$

In this case, the forms of the similarity variable and the similarity solution are

$$\zeta = \frac{r}{z},\ \phi(r,z) = F(\zeta),\ A(r,z) = G(\zeta).$$

Using these substitutions in Eqs. (6) and (7), we obtain the following reduced system of ODEs

$$\frac{F''(\zeta)}{F(\zeta)}(1+\zeta^2) - \frac{F'^2(\zeta)}{F^2(\zeta)}(1+\zeta^2) + \frac{F'(\zeta)}{\zeta F'(\zeta)} \times (1+2\zeta^2) - 1 - \frac{1}{2}F^2\zeta G'^2(\zeta)(1+\zeta^2) = 0, \quad (29)$$

$$G''(\zeta)(1+\zeta^2) - \frac{G'(\zeta)}{\zeta} + \frac{2F'(\zeta)G'(\zeta)}{F(\zeta)}(1+\zeta^2). \quad (30)$$

By solving this system of ODEs, we have

$$F(\zeta) = \frac{\sqrt{(2)}c_2\zeta\left(2c_1^2 + \exp 2\left(\dfrac{\tanh^{-1}\left(1/\sqrt{1+\zeta^2}\right)}{\sqrt{2}c_2}\right)\exp\left(\dfrac{2c_3}{\sqrt{2}c_2}\right)\right)}{4\exp\left(\dfrac{\tanh^{-1}\left(1/\sqrt{1+\zeta^2}\right)}{\sqrt{2}c_2}\right)\exp\left(\dfrac{c_3}{\sqrt{2}c_2}\right)}, \quad (31)$$

$$G(\zeta) = \frac{4\sqrt{2}c_1}{c_2\left(2c_1^2 + \exp\left(\dfrac{\left(\tanh^{-1}\left(1/\sqrt{1+\zeta^2}\right) + c_3\right)\sqrt{2}}{c_2}\right)\right)}. \quad (32)$$

Hence, the solution for nonlinear system (6)–(9) is given by

$$\phi(r,z) = \log\left(z\frac{\sqrt{(2)}c_2\left(\dfrac{r}{z}\right)\left(2c_1^2 + \left(\exp\left(\dfrac{\tanh^{-1}\left(\dfrac{1}{\sqrt{1+r/z^2}}\right)}{\sqrt{2}c_2}\right)\right)^2\exp\left(\dfrac{2c_3}{\sqrt{2}c_2}\right)\right)}{4\exp\left(\dfrac{\tanh^{-1}\left(1/\sqrt{1+r/z^2}\right)}{\sqrt{2}c_2}\right)\exp\left(\dfrac{c_3}{\sqrt{2}c_2}\right)}\right), \quad (33)$$

$$A(r,z) = \frac{4\sqrt{2}c_1}{c_2\left(2c_1^2 + \exp\left(\dfrac{\left(\tanh^{-1}\left(\dfrac{1}{\sqrt{1+r/z^2}}\right) + c_3\right)\sqrt{2}}{c_2}\right)\right)}, \quad (34)$$

$$\Lambda(r,z) = -\frac{1}{2c_2^2}\left(\int p(r,z)\,\mathrm{d}r - 2\int q(r,z)\,\mathrm{d}z\right), \quad (35)$$

where $p(r,z) = \dfrac{l(r,z)}{m(r,z)}$,

$$l(r,z) = \left(-z^2 + 2c_2^2r^2 + 2\sqrt{2}z^2c_2\sqrt{\frac{r^2+z^2}{z^2}} + r^2 - 2c_2^2z^2\right)j(r,z),$$

$$-\ 4c_2^2c_1^2z^2 - 4c_1^2c_2^2r^2 + 2c_1^2r^2 - 4\sqrt{2}c_1^2c_2z^2\sqrt{\frac{r^2+z^2}{z^2}} - 2c_1^2z^2,$$

$$m(r,z) = r(r^2+z^2)(2c_1^2 + j(r,z)),$$

$$q(r,z) = \frac{\left(\left(\sqrt{2}c_2\sqrt{\frac{r^2+z^2}{z^2}} - 1\right) j(r,z) - 2c_1^2 - 2\sqrt{2}c_2c_1^2\sqrt{\frac{r^2+z^2}{z^2}}\right) z}{(r^2+z^2)(2c_1^2 + j(r,z))},$$

$$j(r,z) = \exp\left(\frac{\tanh^{-1}\left(1/\sqrt{r^2+z^2/z^2} + c_3\right)\sqrt{2}}{c_2}\right),$$

with $c_1, c_2$, and $c_3$ being arbitrary constants.

### 3.3. Vector field $V_2 + V_1$

For this vector field, the forms of the similarity variable and the similarity solution are

$$\zeta = r,\ \phi(r,z) = F(\zeta),\ A(r,z) = z + G(\zeta).$$

Using these substitutions, we can reduce Eqs. (6) and (7) to

$$F''(\zeta) + \frac{F'(\zeta)}{\zeta} - \frac{\mathrm{e}^{2F(\zeta)}(1+G'(\zeta)^2)}{2\zeta^2} = 0, \quad (36)$$

$$G''(\zeta) - \frac{G'(\zeta)}{\zeta} + 2F'(\zeta)G'(\zeta) = 0. \quad (37)$$

By solving the second ODE, we obtain

$$G'(\zeta) = c_1\zeta\,\mathrm{e}^{-2F(\zeta)}. \quad (38)$$

By substituting this $G'(\zeta)$ into the first ODE, we obtain

$$F''(\zeta) + \frac{F'(\zeta)}{\zeta} - \frac{\mathrm{e}^{2F(\zeta)}(1 + c_1^2\zeta^2\,\mathrm{e}^{-4F(\zeta)})}{2\zeta^2} = 0, \quad (39)$$

where $c_1$ is an arbitrary constant. It is quite difficult to solve this ODE.

### 3.4. Vector field $V_3$

Corresponding to this vector field, we only have a constant solution.

### 3.5. Vector field $V_3 + V_1$

For this vector field, the forms of the similarity variable and the similarity solution are

$$\zeta = r,\ \phi(r,z) = -z + F(\zeta),\ A(r,z) = \mathrm{e}^z G(\zeta).$$

Using these substitutions, we reduce Eqs. (6) and (7) to

$$2\zeta^2F''(\zeta) + 2\zeta F'(\zeta) - \mathrm{e}^{2F(\zeta)}(G'^2(\zeta) + G^2(\zeta)) = 0, \quad (40)$$

$$\zeta G''(\zeta) - G'(\zeta) - \zeta G(\zeta) + 2\zeta F'(\zeta)G'(\zeta) = 0. \quad (41)$$

After the substitution of $G(\zeta) = \mathrm{e}^{-F(\zeta)}H(\zeta)$ and $F'(\zeta) = M(\zeta)$, we have

$$2\zeta^2M'(\zeta) + 2\zeta M(\zeta) - M(\zeta)^2H(\zeta)^2 + 2M(\zeta)H(\zeta)H'(\zeta) - H'(\zeta)^2 - H(\zeta)^2 = 0, \quad (42)$$

$$\zeta M'(\zeta)H(\zeta) + \zeta M(\zeta)^2H(\zeta) - \zeta H''(\zeta) - M(\zeta)H(\zeta) + H'(\zeta) + \zeta H(\zeta) = 0. \quad (43)$$

For the solution of the above ODEs, three cases are considered depending on $H(\zeta)$. In the first case, $H(\zeta) = 0$, it is not physically interesting. In the second and the third cases, $H(\zeta) = \pm\sqrt{2}\iota\zeta$, the above ODEs are reduced into a single equation

$$\zeta^2M'(\zeta) - \zeta M(\zeta) + \zeta^2M(\zeta)^2 + 1 + \zeta^2 = 0. \quad (44)$$

The solution for the above ODE is

$$M(\zeta) = \frac{-c_1\zeta\mathrm{Y}_1(\zeta) + \mathrm{J}_0(\zeta) - \zeta\mathrm{J}_1(\zeta) + c_1\mathrm{Y}_0(\zeta)}{\zeta(c_1\mathrm{Y}_0(\zeta) + \mathrm{J}_0(\zeta))}. \quad (45)$$

Then the solution of Eqs. (6) and (7) is

$$\phi(r,z) = -z + \int \frac{-c_1r\mathrm{Y}_1(r) + \mathrm{J}_0(r) - r\mathrm{J}_1(r) + c_1\mathrm{Y}_0(r)}{r(c_1\mathrm{Y}_0(r) + \mathrm{J}_0(r))}\,\mathrm{d}r, \quad (46)$$

$$A(r,z) = \mathrm{e}^z \exp\left(-\int \frac{-c_1r\mathrm{Y}_1(r) + \mathrm{J}_0(r) - r\mathrm{J}_1(r) + c_1\mathrm{Y}_0(r)}{r(c_1\mathrm{Y}_0(r) + \mathrm{J}_0(r))}\,\mathrm{d}r\right) \pm\sqrt{2}\iota r, \quad (47)$$

$$\Lambda(r,z) = \log(r) + 2\log(c_1\mathrm{Y}_0(r) + \mathrm{J}_0(r)) - 2z, \quad (48)$$

where $c_1$ is an arbitrary constant, $\mathrm{J}_v(r)$ and $\mathrm{Y}_v(r)$ are the modified Bessel functions of the first and the second kinds, respectively, and satisfy the modified Bessel equation

$$r^2\mathrm{Y}'' + r\mathrm{Y}' - (r^2 + v^2)\mathrm{Y} = 0. \tag{49}$$

### 3.6. Vector field $V_4 + \alpha V_3$

When $\alpha = 1$, the forms of the similarity variable and the similarity solution are

$$\zeta = \frac{r}{z},\ \phi(r,z) = F(\zeta),\ A(r,z) = zG(\zeta).$$

By using these in Eqs. (6) and (7), we obtain the system of reduced ODEs

$$F''(\zeta)(1+\zeta^2) + \frac{F'(\zeta)}{\zeta}(1+2\zeta^2) - \frac{1}{2}\,\mathrm{e}^{2F(\zeta)} \frac{\Big(G'(\zeta)^2(1+\zeta^2) + G(\zeta)^2 - 2\zeta G(\zeta)G'(\zeta)\Big)}{\zeta^2} = 0, \tag{50}$$

$$G''(\zeta)(1+\zeta^2) - \frac{G'(\zeta)}{\zeta} + 2F'(\zeta)G'(\zeta)(1+\zeta^2) - 2\zeta F'(\zeta)G(\zeta) = 0. \tag{51}$$

After the substitution of $G(\zeta) = \mathrm{e}^{-F(\zeta)}H(\zeta)$ and $F'(\zeta) = M(\zeta)$, we obtain

$$-2\zeta^2 M'(\zeta) - 2\zeta M(\zeta) - 2\zeta^4 M'(\zeta) - 4\zeta^3 M(\zeta) + M(\zeta)^2 H(\zeta)^2(1+\zeta^2) + H'(\zeta)^2(1+\zeta^2) \tag{52}$$

$$-2M(\zeta)H(\zeta)H'(\zeta)(1+\zeta^2) + H(\zeta)^2 + 2\zeta M(\zeta)H(\zeta)^2 - 2\zeta H(\zeta)H'(\zeta) = 0, \tag{53}$$

$$\zeta M'(\zeta)H(\zeta) + \zeta M(\zeta)^2 H(\zeta) - \zeta H''(\zeta) + \zeta^3 M'(\zeta)H(\zeta) - \zeta^3 H''(\zeta) - M(\zeta)H(\zeta) + H'(\zeta) + 2\zeta^2 M(\zeta)H(\zeta) - \zeta^3 M(\zeta)^2 H(\zeta) = 0. \tag{54}$$

## 4. Discussion and conclusion

We have derived new exact solutions for Einstein–Maxwell equations in general relativity corresponding to the magnetostatic fields. The obtained solutions describe the external gravitational fields of rotating bodies, such as stars and galaxies. The symmetry method based on the Fréchet derivatives is used to obtain new exact analytic solutions for the nonlinear system of PDEs. We exploit the symmetries of Einstein–Maxwell equations to derive some ansatz leading to the reduction of variables, with which the analytic solutions are easier to obtain by considering the optimal system of conjugacy inequivalent subgroups. The solutions (33)–(35) and (46)–(48) are new exact analytic solutions, which have not been obtained in the literature yet. The obtained solutions depend on both independent variables $r$ and $z$, and may be interesting for further applications.

## References


[1] Lobanov A P 1998 *Astron. Astrophys.* **79** 330
[2] Aharonian F A 2002 *MNRAS* **332** 215
[3] Liu Y T, Shapiro S L and Stephens B C 2007 *Phys. Rev.* D **76** 084017
[4] Vlemmings W H T, Diamond P J and Imai H 2006 *Nature* **440** 58
[5] Aguilera D N, Pons J A and Miralles J A 2008 *Ap. J.* **673** L167
[6] Hennebelle P and Fromang S 2008 *Astron. Astrophys.* **477** 9
[7] Pavlov G G and Bezchastnov V G 2005 *Ap. J.* **635** L61
[8] Ibrahim A I, Swank J H and Parke W 2003 *Ap. J.* **584** L17
[9] Reissner H 1916 *Ann. Phys.* **50** 106
[10] Nordstrom G 1918 *Verhandl. Koninkl. Ned. Akad. Wetenschap. Afdel. Natuurk.* **26** 1201
[11] Bonnor W B 1953 *Proc. Phys. Soc.* A **66** 145
[12] Misra M and Radhakrishna L *Proc. Nat. Inst. Sci. India* A **28** 632
[13] García-Reyes G and Gonjález G A 2007 *Brazilian J. Phys.* **37** 1094
[14] Stephani H, Kramer D, MacCallum M, Hoenselaers D and Herlt E 2003 *Exact Solutions of Einstein Field Equations* (2nd edn.) (Cambridge: Cambridge University Press)
[15] Ji A C, Liu W M, Song J L and Zhou F 2008 *Phys. Rev. Lett.* **101** 010402
[16] Liang Z X, Zhang Z D and Liu W M 2005 *Phys. Rev. Lett.* **94** 050402
[17] Liu W M, Wu B and Niu Q 2000 *Phys. Rev. Lett.* **84** 2294
[18] Singh K and Gupta R K 2006 *Int. J. Eng. Sci.* **44** 1256
[19] Zhao L, Fu J L and Chen B Y 2010 *Chin. Phys.* B **19** 010301
[20] Li Y, Fang J H and Zhang K J 2011 *Chin. Phys.* B **20** 030201
[21] Ma W X and Chen M 2009 *Appl. Math. Comp.* **215** 2835
[22] Goyal N and Gupta R K 2011 *Phys. Scr.* **85** 015004
[23] Bhutani O P, Singh K and Kalra D K 2003 *Int. J. Eng. Sci.* **41** 769
[24] Steinberg S 1979 *Symmetry Methods in Differential Equations. Technical Report No. 367* (Albuquerque: University of New Mexico)
[25] Gandarias M L and Bruzon M S 1998 *J. Nonlinear Math. Phys.* **5** 8
[26] Attallaha S K, El-Sabbagh M F and Ali A T 2007 *Commun. Nonlinear Sci. Numer. Simul.* **12** 1153
[27] Ali A T 2009 *Phys. Scr.* **79** 035006
[28] Bluman G W and Cole J D 1974 *Similarity Methods for Differential Equations* (New York: Springer-Verlag)
[29] Ovsiannikov L V 1982 *Group Analysis of Differential Equations* (New York: Academic Press)