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A Class of Exact Solutions to Navier-Stokes Equations for the Given Vorticity

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Abstract: A class of exact solutions are determined for steady plane motion of an incompressible fluid of variable viscosity with heat transfer. This class consist of flows for which the vorticity distribution is proportional to the stream function perturbed by a exponential stream. Defining a transformation variable, the governing Navier-Stokes Equations are transformed into simple ordinary differential equations and a class of exact solution is obtained. Finally, the influence of the pertinent parameters on the fluid motion are underlined by graphical illustrations.

Keywords: exact solutions; Navier-Stokes equations; steady plane flows; incompressible fluid; variable viscosity; heat transfer and prescribed vorticity

1 Introduction

Navier-Stokes equations, are basic equations of fluid mechanics, are set of non-linear partial differential equations with very small number of exact solutions. At present, numerical solutions to fluid mechanics problems are very attractive due to wide availability of computer programs. But these numerical solutions are insignificant if they can not be compared with either analytical solutions or experimental result. Exact solutions are important not only because they are solution of some fundamental flows, but also because they serve as accuracy standards for approximate methods, whether numerical, asymptotic or experimental.Wang [1] has given an excellent review of these solutions of the Navier-Stokes Equation. These known solutions of viscous incompressible Newtonian fluid may be classified generally into three types:

(i)- Flows for which the non-linear inertial terms in the linear momentum equations vanish identically. Parallel flows and flow with uniform suction are examples of these flows.

(ii)- Similarity properties of the flows such that the flow equations reduce to a set of ordinary differential equations. Stagnation point flow is an example of such flow.

(iii)- Flow for which the vorticity function or stream function is chosen so that the governing equations in terms of the stream function reduce to a linear equation.

By considering the vorticity distribution directly proportional to the stream function $\nabla^2\psi = K\psi$, Taylor [2] showed that the non-linearities are self-canceling and obtained an exact solution which represent the decay of the double array of vortices. Kampe-De-Feriet [3] generalized the Taylor's idea by taking the vorticity of the form $\nabla^2\psi = f(\psi)$. Kovasznay [4] extended Taylor's idea by taking the vorticity to be proportional to the stream function perturbed by a uniform stream of the form $\nabla^2\psi = y + (K^2 - 4\pi^2)\psi$. Kovaszany was able to linearize the Navier-Stokes equation and determine an exact solution for steady flow, which resembles that the downstream of a two-dimensional gird. Wang [5] was also able to linearize the Navier-Stokes equations and showed that the result established Taylor and Kovasznay could be obtained from his finding as special cases by taking the vorticity $\nabla^2\psi = Cy + A\psi$. Lin and Tobak [6], Hui [7] and Naeem and Jamil [8] obtained more results by studing similar flows, taking $\nabla^2\psi = K(\psi - Rz)$, $\nabla^2\psi = K(\psi - Ry)$ and $\nabla^2\psi = K(\psi - Uy)$.

The solutions for Newtonian and non-Newtonian fluid by assuming certain form of vorticity distribution or stream function, are obtained by researcher such as Jeffrey [9], Riabouchinsky [10], Nemenyi [11], Ting [12], Rajagopal [13], Rajagopal and Gupta [14], Siddiqui and Kaloni [15], Wang [16], Benharbit and Siddiqui [17], Chandna and Oku-Ukpong [18], Oku-Ukpong and Chandna [19], Chandna and Oku-Ukpong [20], Scconmandi [21], Labropulu [22], Labropulu [23], Mohyuddin et al. [24] and more recently Islam et al. [25] and Hayat et al. [26].

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In this note, we present a class of exact solutions to the equation governing the steady plane flows of an incompressible fluid with variable viscosity and heat transfer for which the vorticity distribution is proportional to the stream function perturbed by a exponential stream of the form $\nabla^2\psi = K\big(\psi - Ue^{ax+by}\big)$. We point out that the exact solutions obtained by taking this form of vorticity to the best of our knowledge is yet not consider either in Newtonian or non-Newtonian flows.

The plan of this paper is as follows: In section 2 basic flow equations are considered and are transformed into a new system of equations. In section 3, some exact solutions of the new system of equations are determined. The method used in determining the exact solutions to these equations is straightforward.

2 Basic governing equations

The basic non-dimensional equation governing the steady plane flow of an incompressible fluid of variable viscosity, in the absence of external force and with no heat addition from Naeem and Jamil [8] are:

$$u_x + v_y = 0, \tag{1}$$

$$uu_x + vu_y = -p_x + \frac{1}{Re}\Big[(2\mu u_x)_x + \big(\mu(u_y + v_x)\big)_y\Big], \tag{2}$$

$$uv_x + vv_y = -p_y + \frac{1}{Re}\Big[(2\mu v_y)_y + \big(\mu(u_y + v_x)\big)_x\Big], \tag{3}$$

$$uT_x + vT_y = \frac{1}{RePr}(T_{xx} + T_{yy}) + \frac{Ec}{Re}\mu\Big[2(u_x^2 + v_y^2) + (u_y + v_x)^2\Big], \tag{4}$$

where u, v are the velocity components, p the pressure, T the temperature, μ the viscosity, Re, Pr and Ec are Reynolds, Prandtl and Eckert numbers respectively. Eq. (1) implies the existence of the stream function ψ such that

$$u = \psi_y,\ v = -\psi_x. \tag{5}$$

The system of Eqs.(1-4) on utilizing Eq.(5), transform into the following system of equations:

$$L_x = -\psi_x\omega + \frac{1}{Re}\Big[\mu(\psi_{yy} - \psi_{xx})\Big]_y, \tag{6}$$

$$L_y = -\psi_y\omega + \frac{1}{Re}\Big[\mu(\psi_{yy} - \psi_{xx})\Big]_x - \frac{4}{Re}(\mu\psi_{xy})_y, \tag{7}$$

$$\psi_y T_x - \psi_x T_y = \frac{1}{RePr}(T_{xx} + T_{yy}) + \frac{Ec}{Re}\mu\Big[4(\psi_{xy})^2 + (\psi_{yy} - \psi_{xx})^2\Big], \tag{8}$$

where the vorticity function ω and the generalized energy function L are defined by

$$\omega = -(\psi_{xx} + \psi_{yy}), \tag{9}$$

$$L = p + \frac{1}{2}(\psi_x^2 + \psi_y^2) - \frac{2\mu\psi_{xy}}{Re}. \tag{10}$$

Once a solution of system of Eqs.(6-8) is determined, the pressure p is obtained form Eq.(10). We shall investigate fluid motion for which the vorticity distribution is proportional to the stream function perturbed by a exponential stream. Therefore we set

$$\psi_{xx} + \psi_{yy} = K\big(\psi - Ue^{ax+by}\big), \tag{11}$$

where $K, a, b \neq 0$, $a \neq b$ and U are real constants. On substituting

$$\Psi = \psi - Ue^{ax+by}, \tag{12}$$

and employing Eq.(11), the Eq.(9) becomes

$$\omega = -K\Psi. \tag{13}$$

Equations (6) and (7), utilizing Eqs.(12) and (13), become

$$L_x = \Big(\frac{K\Psi^2}{2}\Big)_x + aUK\Psi e^{ax+by} + \frac{1}{Re}\Big[\mu\big(\Psi_{yy} - \Psi_{xx} + (b^2 - a^2)Ue^{ax+by}\big)\Big]_y, \tag{14}$$

$$L_y = \Big(\frac{K\Psi^2}{2}\Big)_y + bUK\Psi e^{ax+by} + \frac{1}{Re}\Big[\mu\big(\Psi_{yy} - \Psi_{xx} + (b^2 - a^2)Ue^{ax+by}\big)\Big]_x -$$

$$-\frac{4}{Re}\Big[\mu\big(\Psi_{xy} + abUe^{ax+by}\big)\Big]_y. \tag{15}$$

Equations (14) and (15), on using the integrability condition $L_{xy} = L_{yx}$, provide

$$H_{xx} - H_{yy} + UK(b\Psi_x - a\Psi_y)e^{ax+by} - \frac{4}{Re}\Big[\mu\big(\Psi_{xy} + abUe^{ax+by}\big)\Big]_{xy} = 0, \tag{16}$$

where

$$H = \frac{\mu\big(\Psi_{yy} - \Psi_{xx} + (b^2 - a^2)Ue^{ax+by}\big)}{Re}.$$

Equation(16) is the equation that must be satisfied by the function Ψ and the viscosity μ for the motion of an steady incompressible fluid of variable viscosity in which the vorticity distribution is proportional to the stream function perturbed by a exponential stream. Equation(8), employing Eq.(12), becomes

$$\big(\Psi_y + bUe^{ax+by}\big)T_x - \big(\Psi_x + aUe^{ax+by}\big)T_y \quad = \quad \frac{1}{RePr}(T_{xx} + T_{yy}) +$$

$$+ \frac{Ec}{Re}\mu\Big[4\Big(\Psi_{xy} + abUe^{ax+by}\Big)^2 + \Big(\Psi_{yy} - \Psi_{xx} + (b^2 - a^2)Ue^{ax+by}\Big)^2\Big]. \tag{17}$$

Equation(11), employing Equation(12), becomes

$$\Psi_{xx} + \Psi_{yy} - K\Psi = -(a^2 + b^2)Ue^{ax+by}. \tag{18}$$

Introducing the transformation variable as

$$\xi = ax + by.$$

Transforming the Equations (16), (17) and (18), into new independent variable ξ we have

$$\Psi_{\xi\xi} - \Lambda\Psi = -Ue^{\xi}, \tag{19}$$

where

$$\Lambda = \frac{K}{a^2 + b^2},$$

and

$$\big(\mu\Psi\big)_{\xi\xi} = 0, \tag{20}$$

$$T_{\xi\xi} + EcPr\Lambda^2(a^2 + b^2)\mu\Psi^2 = 0, \tag{21}$$

3 Exact solutions

In this section we present some exact solutions of the system of Equations(19-21) as follows: We consider the following three cases:
Case-I: $\Lambda = -n^2, n > 0$
Case-II: $\Lambda = m^2, m > 0$
Case-III: $\Lambda = 0$

We now consider these cases separately and determine the solution of the Equations(19-21). Our strategy is that first we find Ψ from equation(19) and use this Ψ to determine μ, T, ψ, u, v and p from system of Eqs.(20), (21), (12), (5) and (10).

Case-I

For this case the solution of Eq.(19) in the physical plane is given by

$$\Psi = A_1 \cos(n(ax + by) + A_2) - \frac{Ue^{ax+by}}{n^2 + 1}, \tag{22}$$

where A_1 and A_2 are real constants. Equation(20), utilizing Eq.(22), gives

$$\mu = \frac{A_3(ax + by) + A_4}{A_1 \cos(n(ax + by) + A_2) - \frac{Ue^{ax+by}}{n^2+1}}, \tag{23}$$

where A_3 and A_4 are real constants. Equation(21), using Eq.(23), becomes

$$T_{\xi\xi} + EcPrn^4(a^2 + b^2)(A_3\xi + A_4)\Psi = 0. \tag{24}$$

The solution of Eq.(24) is

$$\begin{aligned} T = \ & \tfrac{EcPrn(a^2+b^2)}{n^2+1} \Bigg[A_3 \Big\{ Un^3 e^{ax+by} \Big(ax + by - 2 \Big) \\ & + A_1(n^2 + 1) \Big\{ n(ax + by) \cos(n(ax + by) + A_2) - 2 \sin(n(ax + by) + A_2) \Big\} \Big\} \\ & + nA_4 \Big\{ Un^2 e^{ax+by} + A_1(n^2 + 1) \cos(n(ax + by) + A_2) \Big\} \Bigg] + A_5(ax + by) + A_6, \end{aligned} \tag{25}$$

where A_5 and A_6 are real constants. The stream function ψ for this case is given by

$$\psi = \frac{Un^2}{n^2 + 1} e^{ax+by} + A_1 \cos(n(ax + by) + A_2). \tag{26}$$

It represent a exponential stream $\frac{Un^2}{n^2+1} e^{ax+by}$ in the positive x-direction plus a perturbation that is periodic in x and y. The component of velocity distribution from Eqs.(5) and (26), and pressure from Eq.(10), are given by

$$u = \frac{Ubn^2}{n^2 + 1} e^{ax+by} - A_1 nb \sin(n(ax + by) + A_2), \tag{27}$$

$$v = -\frac{Uan^2}{n^2 + 1} e^{ax+by} + A_1 na \sin(n(ax + by) + A_2), \tag{28}$$

$$\begin{aligned} p = \ & \tfrac{Un^2(a^2+b^2)}{2(n^2+1)} e^{ax+by} \Bigg[Ue^{ax+by} - 2A_1 \Big\{ \cos(n(ax + by) + A_2) \\ & + n \sin(n(ax + by) + A_2) \Big\} \Bigg] - \tfrac{A_3 n^2 (b^2 - a^2)}{Re} (bx + ay) + \tfrac{2abn^2}{Re} \Big(A_3(ax + by) + A_4 \Big) \\ & - \tfrac{n^2(a^2+b^2)}{2(n^2+1)^2} \Big[Une^{ax+by} - A_1(n^2 + 1) \sin(n(ax + by) + A_2) \Big]^2 + A_7, \end{aligned} \tag{29}$$

where A_7 is real constant.

Case-II

For this case

$$\Psi = B_1 e^{m(ax+by)} + B_2 e^{-m(ax+by)} + \frac{U}{m^2 - 1} e^{ax+by}, \tag{30}$$

where B_1 and B_2 are real constants. Equation(20), utilizing Eq.(30), gives

$$\mu = \frac{B_3(ax+by)+B_4}{B_1e^{m(ax+by)}+B_2e^{-m(ax+by)}+\frac{U}{m^2-1}e^{ax+by}}, \tag{31}$$

where B_3 and B_4 are real constants. Equation(21), using Eq.(31), becomes

$$T_{\xi\xi}+EcPrm^4(a^2+b^2)(B_3\xi+B_4)\Psi=0. \tag{32}$$

The solution of Eq.(32) is

$$\begin{aligned} T = \; & \tfrac{EcPrm(a^2+b^2)}{1-m^2}\Big[B_3\Big\{Um^3e^{m(ax+by)}\Big(ax+by-2\Big)+B_1(m^2-1)\big(m(ax+by)-2\big)e^{m(ax+by)} \\ & +B_2(m^2-1)\big(m(ax+by)+2\big)e^{-m(ax+by)}\Big\}+mB_4\Big\{Um^2e^{m(ax+by)} \\ & +B_1(m^2-1)e^{m(ax+by)}+B_2(m^2-1)\Big\}\Big]+B_5(ax+by)+B_6, \end{aligned} \tag{33}$$

where B_5 and B_6 are real constants. For this case stream function

$$\psi = \frac{Um^2}{m^2-1}e^{ax+by}+B_1e^{m(ax+by)}+B_2e^{-m(ax+by)}, \tag{34}$$

represent a exponential stream $\frac{Um^2}{m^2-1}e^{ax+by}$ in the positive x-direction plus a perturbation that is not periodic in x and y. The components of velocity distribution and pressure, in this case are given by

$$u = \frac{Ubm^2}{m^2-1}e^{ax+by}+B_1mbe^{m(ax+by)}-B_2mbe^{-m(ax+by)}, \tag{35}$$

$$v = -\frac{Uam^2}{m^2-1}e^{ax+by}-B_1mae^{m(ax+by)}+B_2mae^{-m(ax+by)}, \tag{36}$$

$$\begin{aligned} p = \; & \tfrac{Um^2(a^2+b^2)}{2(m^2-1)}\Big[Ue^{2(ax+by)}+2B_1(m-1)e^{(m+1)(ax+by)}-2B_2(m+1)e^{(1-m)(ax+by)}\Big] \\ & +\tfrac{B_3m^2(b^2-a^2)}{Re}(bx+ay)-\tfrac{2abm^2}{Re}\big(B_3(ax+by)+B_4\big) \\ & -\tfrac{m^2(a^2+b^2)}{(m^2-1)^2}e^{-2m(ax+by)}\Big[Ume^{(1+m)(ax+by)}+B_1(m^2-1)e^{2m(ax+by)}-B_2(m^2-1)\Big]^2+B_7, \end{aligned} \tag{37}$$

where B_7 is real constant.

Case-III

For this case, we have

$$\Psi = C_1(ax+by)+C_2-Ue^{ax+by}, \tag{38}$$

$$\mu = \frac{C_3(ax+by)+C_4}{C_1(ax+by)+C_2-Ue^{ax+by}}, \tag{39}$$

$$T = C_5(ax+by)+C_6, \tag{40}$$

where C_1, C_2,..., C_6 are real constants. $\Lambda=0$, corresponds to an irrotational flow and it is the following uniform flow

$$\psi = C_1(ax+by)+C_2. \tag{41}$$

The components of velocity distribution and pressure, in this case are

$$u = C_1b, \tag{42}$$

$$v = -C_1a, \tag{43}$$

$$p = -\frac{(a^2+b^2)C_1^2}{2}+C_7, \tag{44}$$

where C_7 is real constant.

4 Results and discussion

This section deals with the effect of the pertinent parameters n, m, a and b on the components of velocity profiles $v(x, y)$ and $u(x, y)$. Figures(1-4) are for case-I and figures(5-8) are for case-II. The effect of Prandtl number Pr and Eckert number Ec on the temperature profile is also discussed. Figs. 1 and 2 show the influence on components of velocity profile v and u in the direction of y and x respectively, for different values of parameter n. It is clear form these figures, as it is expected that velocity components have oscillating behavior and both velocity components are increasing function of n in absolute value. Furthermore, magnitude of velocity components $v(x, y)$ and $u(x, y)$ or amplitude of the oscillation increase with increase of parameter n. It is noted that increase in this parameter increase the velocity components in the most narrow position. Similar effects are observed in the direction of y and x for the parameter b and a on the velocity components $v(x, y)$ and $u(x, y)$, respectively as shown in Figs. 3 and 4. Figs. 5, 6, 7 and 8 show the influence of parameter m, a and b on the velocity components $v(x, y)$ and $u(x, y)$, in the direction of y and x respectively. It is clear from these figure that both velocity components are increasing function of these parameters in absolute value.

Finally, the influence of Prandtl and Eckert numbers on the temperature form Eqs. (25) and (33) are clear that temperature is directly proportional to the Prandtl and Eckert numbers and temperature increase or decrease with increase or decrease of Prandtl and Eckert numbers.

5 Conclusions

A class of exact solutions of the equations governing the steady plane motion of an incompressible fluid with variable viscosity and heat transfer is determined. This class consist of flows for which the vorticity distribution is proportional to the stream function perturbed by a exponential stream.

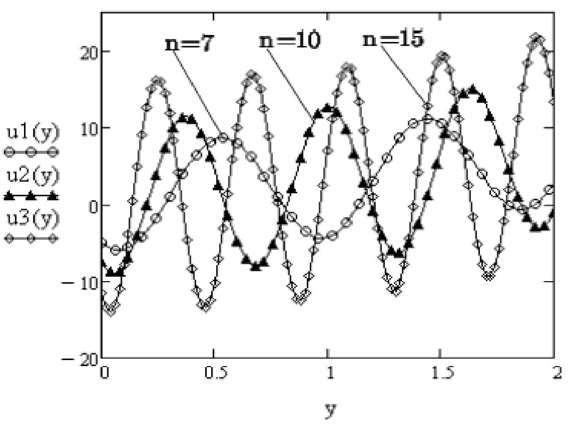

Figure 1: Profile of the velocity component u given by Eq. (27) in the direction of y, for $A_1 = A_2 = U = b = 1, a = 0$ and different values of n.

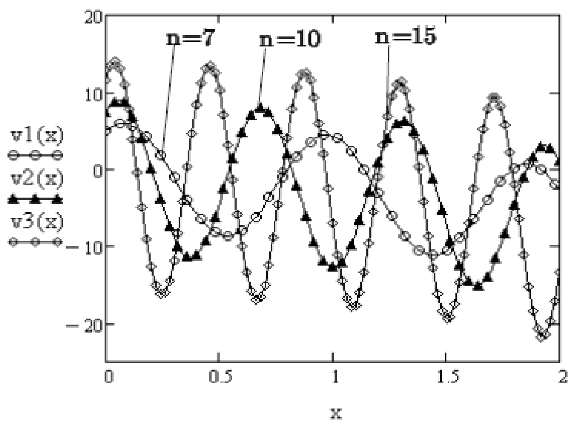

Figure 2: Profile of the velocity component v given by Eq. (28) in the direction of x, for $A_1 = A_2 = U = b = 1, b = 0$ and different values of n.

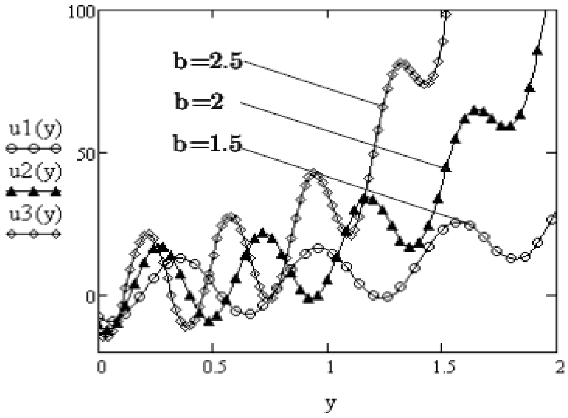

Figure 3: Profile of the velocity component u given by Eq. (27) in the direction of y, for $A_1 = A_2 = U = 1, n = 7, a = 0$ and different values of b.

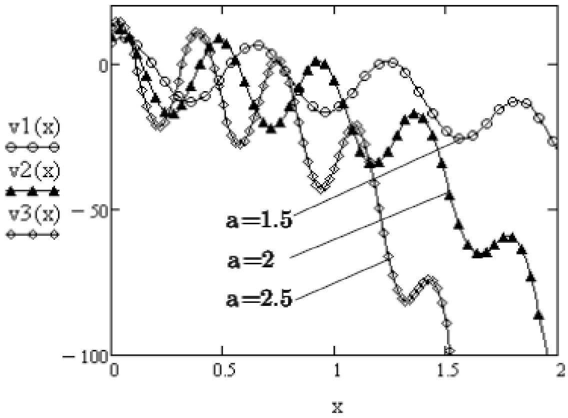

Figure 4: Profile of the velocity component v given by Eq. (28) in the direction of x, for $A_1 = A_2 = U = 1, n = 7, b = 0$ and different values of a.

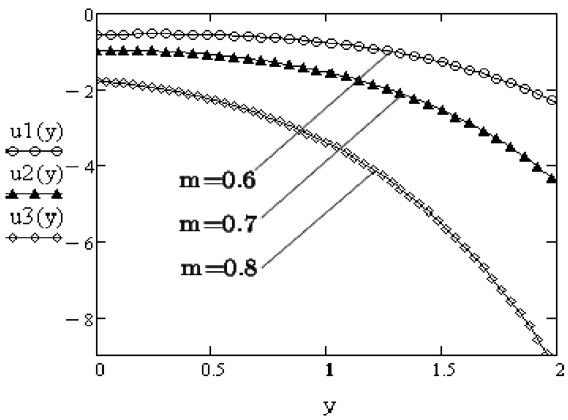

Figure 5: Profile of the velocity component u given by Eq. (35) in the direction of y, for $A_1 = A_2 = U = b = 1, a = 0$ and different values of m.

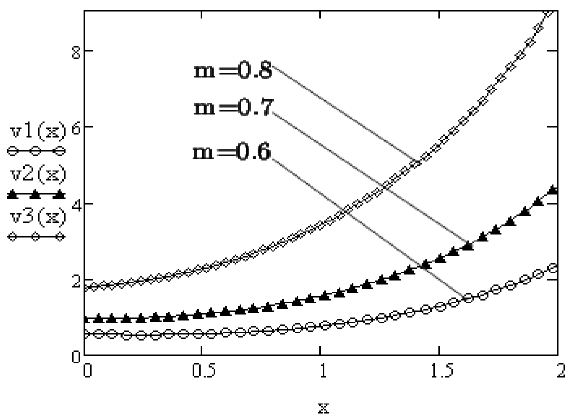

Figure 6: Profile of the velocity component v given by Eq. (36) in the direction of x, for $A_1 = A_2 = U = b = 1, b = 0$ and different values of m.

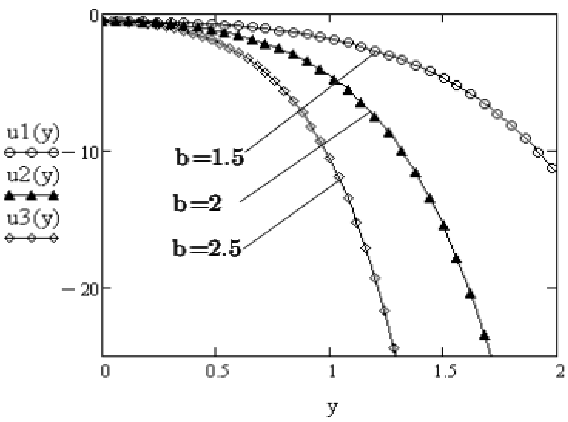

Figure 7: Profile of the velocity component u given by Eq. (35) in the direction of y, for $A_1 = A_2 = U = 1, m = 7, a = 0$ and different values of b.

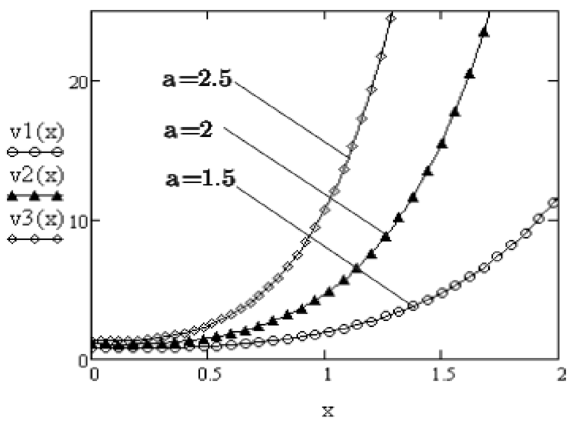

Figure 8: Profile of the velocity component v given by Eq. (36) in the direction of x, for $A_1 = A_2 = U = 1, m = 7, b = 0$ and different values of a.

In order to determine the exact solutions, the flow equations are first written in terms of the stream function ψ, the vorticity function ω and the generalized energy function L. Employing the integrability condition on the generalized energy function L, an equation is determined that must be satisfied by the function Ψ and the viscosity μ for the flow under consideration.

The solutions are obtained through the procedure described in section 3. The solutions in case-I represents a exponential stream plus a perturbation that is periodic in x and y. The solution in case-II, in general, represents a exponential stream plus a perturbation that is not periodic in x and y. When $B_1 = 0$, in the solution of case-II, the solution represents a exponential flow in the region $x > 0$, $y > 0$ perturbed by a part which decays and grows exponentially as x , y increase for $m > 0$ and $m < 0$, respectively. Similarly, we can give description for flow in other regions. Finally the influence of various parameters of interest on the velocity components are plotted and discussed.

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