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1 Full derivation of the Schwarzschild solution

The goal of this document is to provide a full, thoroughly detailed derivation of the Schwarzschild solution. Much of the differential geometric foundations can be found elsewhere (and may be added at a later date). For details on how to get the form of the Reimann curvature tensor and the stress-energy tensor, see the other notes.

Here we begin with a statement of the question.

Question: What is the metric solution for spacetime exterior to a spherically symmetric, static body of radius R and mass M?

To begin answering this problem we will examine the metric of spacetime in spherical coordinates and in a general form. We will work with a west coast signature (+ - - -) and set $c = 1$ for clarity. We will reinsert c at the end. The flat metric in spherical coordinates is expressed as follows

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

Where we can express the non-zero components of the metric tensor as $g_{00} = 1, g_{11} = -1, g_{22} = -r^2, g_{33} = -r^2\sin^2\theta$. As we proceed we will use Greek indices to freely range over 0 through 3 $(0 = t, 1 = r, 2 = \theta, 3 = \phi)$ we will later use Latin indices (i, j, k) to refer to only spatial indices (r, θ, ϕ).

We now generalize this to a more general form,

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu = U dt^2 - V dr^2 - W r^2 d\theta^2 - X r^2 \sin^2\theta d\phi^2$$

The functions U, V, W, X are to be determined but can be limited by the properties of our solution:

1. Spherical symmetry. The functions should not depend on θ of ϕ. Also, the angular part of the metric should have $W = X$. In fact, we can limit this to $W = X = 1$ without loss of generality.
2. Static. The functions are not functions of time. Also, any derivatives with respect to time of any metric component vanishes. Thus $U = U(r), V = V(r)$.
3. Vacuum solution. Outside of the object that is the source of curvature there is no matter or energy. Thus the form of Einstein's solution will simplify a lot in the next section.

Thus are limited general metric is ,

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu = U(r) dt^2 - V(r) dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{1}$$

The metric components are explicitly,

$$g_{00} = U, \quad g_{11} = -V, \qquad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta$$

The metric tensor is symmetric and has the following property,

$$g^\mu_\mu = \sum_\mu \sum_\nu g^{\mu\nu} g_{\mu\nu} \equiv g^{\mu\nu} g_{\mu\nu} = 4. \tag{2}$$

Where in the last expression we introduced the *Einstein summation convention*. Whenever two indices are repeated (one up, one down) in an expression they are to be summed over all dimensions. In general,

$$F^{\alpha\beta\mu}{}_{\lambda\beta} G_{\gamma\mu} \equiv \sum_{\beta=0}^{3} \sum_{\mu=0}^{3} F^{\alpha\beta\mu}{}_{\lambda\beta} G_{\gamma\mu} = (FG)^\alpha{}_{\lambda\gamma}$$

Thus we see we can find the metric tensor with raised indices to be just the inverse of (2),

$$g^{00} = \frac{1}{U}, \quad g^{11} = -\frac{1}{V}, \qquad g^{22} = -\frac{1}{r^2}, \quad g^{33} = -\frac{1}{r^2\sin^2\theta}$$

1.1 Einstein's equation

The goal is to find a solution of Einstein's equation for our metric (1),

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \quad \text{\{ Missing R in the -1/2guv term see eqn (7) \}} \tag{3}$$

FIrst some terminology: $R_{\mu\nu}$ *Ricci tensor*, R *Ricci scalar*, and $T_{\mu\nu}$ *stress-energy tensor* (the last term will vanish for the Schwarzschild solution). Before we can proceed we need to introduce some quantities and unravel the expression above.

The Ricci tensor and scalar in terms of the Reimann curvature tensor

The Ricci tensor and scalar are obtained from the *Reimann curvature tensor*, $R^{\beta}{}_{\nu\rho\sigma}$ that is introduced in the other set of notes. The Ricci tensor is a contraction of the full curvature tensor,

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\nu\alpha}$$

The Ricci scalar is a contraction of the Ricci tensor,

$$R \equiv R^{\mu}{}_{\mu} = g^{\mu\alpha}R_{\mu\alpha}$$

The full Reimann curvature tensor is most compactly described in terms of the *Christoffel symbols* (of the second kind), $\Gamma^{\beta}{}_{n\sigma}$, which will be defined in terms of the metric afterwards.

$$R^{\beta}{}_{\nu\rho\sigma} = \; = \Gamma^{\beta}{}_{\nu\sigma,\rho} - \Gamma^{\beta}{}_{\nu\rho,\sigma} + \Gamma^{\alpha}{}_{\nu\sigma}\Gamma^{\beta}{}_{\alpha\rho} - \Gamma^{\alpha}{}_{\nu\rho}\Gamma^{\beta}{}_{\alpha\sigma} \tag{4}$$

In this expression a short hand notation for partial derivatives is employed. Any index offset by a comma is meant to signify a derivative with respect to that coordinate. I.e. $\Gamma^{\beta}{}_{\nu\sigma,\rho} \equiv \frac{\partial}{\partial x^{\rho}}\Gamma^{\beta}{}_{\nu\sigma} = \partial_{\rho}\Gamma^{\beta}{}_{\nu\sigma}$.

The Christoffel symbols are expressed in terms of the metric tensor,

$$\Gamma^{\mu}{}_{\nu\sigma} = \frac{1}{2}g^{\mu\lambda}\left\{g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\nu\sigma,\lambda}\right\} \tag{5}$$

We now see what needs to be done. We have the components of the metric tensor in terms of our functions to be determined, U, V the next step is to find all of the Christoffel symbols. The last step is to find the Ricci tensor and scalar.

1.2 The Christoffel symbols

To simplify the numerous calculations to come we establish some short cuts utilizing the properties of our solution.

1. Any derivatives with respect to time vanish. That is, any term like $g_{\mu\nu,0} = 0$. Thus we can quickly dismiss many terms.
2. All off diagonal terms vanish. That is, any $g_{\mu\nu} = 0$ if $\mu \neq \nu$.
3. The Christoffel symbols are symmetric in the lower two indices (verify for yourself). I.e. $\Gamma^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu}$ thus cutting down on the number of terms significantly.
4. The first term in the general expression of (5) must be on the diagonal. Notice in (5) that the second upper index is to be summed over. However we can set it at the outset because the two upper indices must match, otherwise the whole expression vanishes.
5. Also note that only derivatives of U, V with respect to r (or 1) are non-vanishing.

We will proceed term by term, in order of the upper index and recall Greek indices run over all four values while Latin only run over 1,2,3.

$$\Gamma^{\mu}{}_{\nu\sigma} = \frac{1}{2}g^{\mu\lambda}\left[g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\nu\sigma,\lambda}\right]$$

and our non-vanishing metric components are,

$$\begin{aligned} g_{00} &= U, \quad g_{11} = -V, \qquad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta \\ g^{00} &= \frac{1}{U}, \quad g^{11} = -\frac{1}{V}, \qquad g^{22} = -\frac{1}{r^2}, \quad g^{33} = -\frac{1}{r^2\sin^2\theta} \end{aligned}$$

$\Gamma^0_{\mu\nu}$

$$\begin{aligned} \Gamma^0_{00} &= \frac{1}{2}g^{00}\left[g_{00,0} + g_{00,0} - g_{00,0}\right] = 0 \\ \Gamma^0_{0i} &= \frac{1}{2}g^{00}\left[g_{00,i} + g_{0i,0} - g_{0i,0}\right] = \frac{1}{2}g^{00}\partial_1 g_{00} = \frac{1}{2}\frac{1}{U}\partial_r U \\ \Gamma^0_{ij} &= \frac{1}{2}g^{00}\left[g_{0i,j} + g_{0j,i} - g_{ij,0}\right] = 0 \end{aligned}$$

$\Gamma^1_{\mu\nu}$

$$\begin{aligned} \Gamma^1_{00} &= \frac{1}{2}g^{11}\left[g_{10,0} + g_{10,0} - g_{00,1}\right] = -\frac{1}{2}g^{11}\partial_1 g_{00} = \frac{1}{2}\frac{1}{V}\partial_r U \\ \Gamma^1_{0i} &= \frac{1}{2}g^{11}\left[g_{10,i} + g_{1i,0} - g_{0i,1}\right] = 0 \\ \Gamma^1_{i,j\neq i} &= \frac{1}{2}g^{11}\left[g_{1i,j} + g_{1j,i} - g_{ij,1}\right] = 0 \quad \text{because } V \text{ is a function of } r \text{ only} \\ \Gamma^1_{11} &= \frac{1}{2}g^{11}\left[g_{11,1} + g_{11,1} - g_{11,1}\right] = \frac{1}{2}g^{11}\partial_1 g_{11} = \frac{1}{2}\frac{1}{V}\partial_r V \\ \Gamma^1_{22} &= \frac{1}{2}g^{11}\left[g_{12,2} + g_{12,2} - g_{22,1}\right] = -\frac{1}{2}g^{11}\partial_1 g_{22} = -\frac{1}{2}\frac{1}{V}\partial_r r^2 = -\frac{r}{V} \\ \Gamma^1_{33} &= \frac{1}{2}g^{11}\left[g_{13,3} + g_{13,3} - g_{33,1}\right] = -\frac{1}{2}g^{11}\partial_1 g_{33} = -\frac{1}{2}\frac{1}{V}\partial_r r^2\sin^2\theta = -\frac{r}{V}\sin^2\theta \end{aligned}$$

$\Gamma^2_{\mu\nu}$

$$\begin{aligned} \Gamma^2_{00} &= \frac{1}{2}g^{22}\left[g_{20,0} + g_{20,0} - g_{00,2}\right] = 0 \qquad \Gamma^2_{0i} = \frac{1}{2}g^{22}\left[g_{20,i} + g_{2i,0} - g_{0i,2}\right] = 0 \\ \Gamma^2_{ii} &= \frac{1}{2}g^{22}\left[g_{2i,i} + g_{2i,i} - g_{ii,2}\right] = -\frac{1}{2}g^{22}\partial_2 g_{33} = -\frac{1}{2r^2}r^2\partial_\theta\sin^2\theta = -\cos\theta\sin\theta = \Gamma^2_{33} \\ \Gamma^2_{12} &= \frac{1}{2}g^{22}\left[g_{21,2} + g_{22,1} - g_{12,2}\right] = \frac{1}{2}g^{22}\partial_1 g_{22} = \frac{1}{2r^2}\partial_r r^2 = \frac{1}{r} \\ \Gamma^2_{13} &= \frac{1}{2}g^{22}\left[g_{21,3} + g_{23,1} - g_{13,2}\right] = 0 \\ \Gamma^2_{23} &= \frac{1}{2}g^{22}\left[g_{22,3} + g_{23,2} - g_{23,2}\right] = 0 \end{aligned}$$

$\Gamma^3_{\mu\nu}$

$$\begin{aligned} \Gamma^3_{00} &= \frac{1}{2}g^{33}\left[g_{30,0} + g_{30,0} - g_{00,3}\right] = 0 \qquad \Gamma^3_{0i} = \frac{1}{2}g^{33}\left[g_{30,i} + g_{3i,0} - g_{0i,3}\right] = 0 \\ \Gamma^3_{ii} &= \frac{1}{2}g^{33}\left[g_{3i,i} + g_{3i,i} - g_{ii,3}\right] = 0 \\ \Gamma^3_{i,j\neq i} &= \frac{1}{2}g^{33}\left[g_{3i,j} + g_{3j,i} - g_{ij,3}\right] = \frac{1}{2}g^{33}\left[g_{3i,j} + g_{3j,i}\right] \longrightarrow \\ \Gamma^3_{13} &= = \frac{1}{2}g^{33}\left[g_{31,3} + g_{33,1}\right] = \frac{1}{2}g^{33}\partial_1 g_{33} = \frac{1}{r^2\sin^2\theta}\sin^2\theta\partial_r r^2 = \frac{1}{r} \\ \Gamma^3_{23} &= = \frac{1}{2}g^{33}\left[g_{32,3} + g_{33,2}\right] = \frac{1}{2}g^{33}\partial_2 g_{33} = \frac{1}{r^2\sin^2\theta}\partial_\theta(r^2\sin^2\theta) = \frac{\cos\theta}{\sin\theta} \end{aligned}$$

Summarizing the non-vanishing terms (where prime denotes a derivative with respect to r),

$$\begin{aligned}
\Gamma^0_{01} = \Gamma^0_{10} &= \frac{U'}{2U} \\
\Gamma^1_{00} &= \frac{U'}{2V} \qquad \Gamma^1_{11} = \frac{V'}{2V} \qquad \Gamma^1_{22} = -\frac{r}{V} \qquad \Gamma^1_{33} = -\frac{r}{V}\sin^2\theta \\
\Gamma^2_{12} = \Gamma^2_{21} &= \frac{1}{r} \qquad \Gamma^2_{33} = -\cos\theta\sin\theta \\
\Gamma^3_{13} = \Gamma^3_{31} &= \frac{1}{r} \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta
\end{aligned}$$

1.3 The Ricci tensor

Now that we have the Christoffel symbols we can start to assemble the Ricci tensor and scalar. From equation (4) we can write the Ricci tensor in terms of the Christoffel symbols,

$$\begin{aligned}
R_{\mu\nu} &= R^{\beta}_{\mu\nu\beta} = \Gamma^{\beta}_{\mu\beta,\nu} - \Gamma^{\beta}_{\mu\nu,\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} \\
&= \Gamma^0_{\mu0,\nu} - \Gamma^0_{\mu\nu,0} + \Gamma^\alpha_{\mu0}\Gamma^0_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}\Gamma^0_{\alpha0} \\
&\quad + \Gamma^1_{\mu1,\nu} - \Gamma^1_{\mu\nu,1} + \Gamma^\alpha_{\mu1}\Gamma^1_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}\Gamma^1_{\alpha1} \\
&\quad + \Gamma^2_{\mu2,\nu} - \Gamma^2_{\mu\nu,2} + \Gamma^\alpha_{\mu2}\Gamma^2_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}\Gamma^2_{\alpha2} \\
&\quad + \Gamma^3_{\mu3,\nu} - \Gamma^3_{\mu\nu,3} + \Gamma^\alpha_{\mu3}\Gamma^3_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}\Gamma^3_{\alpha3}
\end{aligned}$$

First we establish the off diagonal terms of the metric tensor.

$R_{\mu\nu}, \mu \neq \nu$

$$\begin{aligned}
R_{0i} &= \Gamma^0_{00,i} - \Gamma^0_{0i,0} + \Gamma^\alpha_{00}\Gamma^0_{\alpha i} - \Gamma^\alpha_{0i}\Gamma^0_{\alpha0} && \longrightarrow 0 - 0 + \Gamma^\alpha_{00}\Gamma^0_{\alpha i} - \Gamma^\alpha_{0i}\Gamma^0_{\alpha0} \\
&\quad + \Gamma^1_{01,i} - \Gamma^1_{0i,1} + \Gamma^\alpha_{01}\Gamma^1_{\alpha i} - \Gamma^\alpha_{0i}\Gamma^1_{\alpha1} && \longrightarrow 0 - 0 + 0 - 0 \\
&\quad + \Gamma^2_{02,i} - \Gamma^2_{0i,2} + \Gamma^\alpha_{02}\Gamma^2_{\alpha i} - \Gamma^\alpha_{0i}\Gamma^2_{\alpha2} && \longrightarrow 0 - 0 + 0 - 0 \\
&\quad + \Gamma^3_{03,i} - \Gamma^3_{0i,3} + \Gamma^\alpha_{03}\Gamma^3_{\alpha i} - \Gamma^\alpha_{0i}\Gamma^3_{\alpha3} && \longrightarrow 0 - 0 + 0 - 0
\end{aligned}$$

All terms vanish since for $\Gamma^\alpha_{00}\Gamma^0_{\alpha i}$ either has $\alpha = 0$, which has the first term vanishing or $\alpha = j$ and i which makes $\Gamma^0_{\mu\nu}$ vanish. Thus $R_{0i} = 0$.

$$\begin{aligned}
R_{i,j\neq i} &= \Gamma^0_{i0,j} - \Gamma^0_{ij,0} + \Gamma^\alpha_{i0}\Gamma^0_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^0_{\alpha0} && \longrightarrow 0 - 0 + \Gamma^\alpha_{i0}\Gamma^0_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^0_{\alpha0} \\
&\quad + \Gamma^1_{i1,j} - \Gamma^1_{ij,1} + \Gamma^\alpha_{i1}\Gamma^1_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^1_{\alpha1} && \longrightarrow 0 - 0 + \Gamma^\alpha_{i1}\Gamma^1_{\alpha j} - 0 \\
&\quad + \Gamma^2_{i2,j} - \Gamma^2_{ij,2} + \Gamma^\alpha_{i2}\Gamma^2_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^2_{\alpha2} && \longrightarrow 0 - 0 + \Gamma^\alpha_{i2}\Gamma^2_{\alpha j} - 0 \\
&\quad + \Gamma^3_{i3,j} - \Gamma^3_{ij,3} + \Gamma^\alpha_{i3}\Gamma^3_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^3_{\alpha3} && \longrightarrow \Gamma^3_{i3,j} - 0 + \Gamma^\alpha_{i3}\Gamma^3_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^3_{\alpha3}
\end{aligned}$$

The terms that are obviously 0 (for one of the 5 reasons stated earlier) are written to the right. Let's examine the remaining terms closely, line by line.

$\Gamma^\alpha_{i0}\Gamma^0_{\alpha j}$. Here if α or $j = 2, 3$ the second symbol vanishes. If $\alpha = 0$ then for this term to be non-vanishing i must equal 1, then j must equal 2 or 3 and the whole term vanishes. If $\alpha = 1$ then i must equal 0 which is not one of the options ($i = 1, 2$, or 3).

$-\Gamma^\alpha_{ij}\Gamma^0_{\alpha0}$ Here we see the only option for α is 1 or else the second term vanishes. Then the first term must vanish since Γ^1_{ij} if $i \neq j$.

$\Gamma^\alpha_{i1}\Gamma^1_{\alpha j}$ Here we see that $\alpha = j$ for the second term to be nonzero. Thus we are left with: $\Gamma^1_{i1}\Gamma^1_{11} + \Gamma^2_{i1}\Gamma^1_{22} + \Gamma^3_{i1}\Gamma^1_{33}$ The first term vanishes because $i \neq j$. The second vanishes because $i = 1$ or 3 and the first symbol vanishes. The third term vanishes because $i = 1$ or 2 for which the first symbol vanishes.

$\Gamma^\alpha_{i2}\Gamma^2_{\alpha j}$ First note that $\alpha \neq 0$ or else the first symbol vanishes. If $\alpha = 1$ then $i = 2$ for the first term to survive but j must be 2 for the second term to survive. This is not possible. If $\alpha = 2$ then $i = 1$ and j must be 1 but this is not possible ($i \neq j$). Lastly, if $\alpha = 3$ then $i = 3$ for the first term to survive but then $j = 3$ for the second term to survive, which it can't. Thus the whole term is 0.

$\Gamma^3_{i3,j}$ This only has $i = 1$ and 2 as non vanishing terms. For the first $j = 2$ or 3 but the symbol is only a function of r, thus those terms vanish. If $i = 2$ then j must be 1 or 3 but the symbol then is only a function of 2 (θ) and it vanishes entirely.

$\Gamma^\alpha_{i3}\Gamma^3_{\alpha j}$ Here we have $\alpha \neq 0$. For $\alpha = 1$ both i and j must be 3 which is not possible. For $\alpha = 2$, then $i = 3$ and $j = 3$ which can't be and the term vanishes. Lastly, for $\alpha = 3$ we have that i is either 1 or 2 and j must be 2 or 1 respectively. Those this term does not vanish and we are left with $\Gamma^3_{13}\Gamma^3_{32} + \Gamma^3_{23}\Gamma^3_{31}$

$-\Gamma^\alpha_{ij}\Gamma^3_{\alpha 3}$ For this term we can only have $\alpha = 1$ or 2. If $\alpha = 1$ then, for the first symbol not to vanish, we must have $i = j$ which is not the case. For $\alpha = 2$ we are left with, $-\Gamma^2_{12}\Gamma^3_{23} - \Gamma^2_{21}\Gamma^3_{23}$

We are left with 4 non vanishing terms. Let's write them out explicitly.

$$\Gamma^3_{13}\Gamma^3_{32} + \Gamma^3_{23}\Gamma^3_{31} - \Gamma^2_{12}\Gamma^3_{23} - \Gamma^2_{21}\Gamma^3_{23} = \frac{1}{r}\cot\theta + \cot\theta\frac{1}{r} - \frac{1}{r}\cot\theta - \frac{1}{r}\cot\theta = 0$$

After all that work we have found out that $R_{\mu\nu} = 0$ for $\mu \neq \nu$.

$R_{\mu\mu}$

$$\begin{aligned}
R_{00} &= \Gamma^0_{00,0} - \Gamma^0_{00,0} + \Gamma^\alpha_{00}\Gamma^0_{\alpha 0} - \Gamma^\alpha_{00}\Gamma^0_{\alpha 0} && \longrightarrow \textit{cancelpairwise} \\
&+\Gamma^1_{01,0} - \Gamma^1_{00,1} + \Gamma^\alpha_{01}\Gamma^1_{\alpha 0} - \Gamma^\alpha_{00}\Gamma^1_{\alpha 1} && \longrightarrow 0 - \Gamma^1_{00,1} + \Gamma^0_{01}\Gamma^1_{00} - \Gamma^1_{00}\Gamma^1_{11} \\
&+\Gamma^2_{02,0} - \Gamma^2_{00,2} + \Gamma^\alpha_{02}\Gamma^2_{\alpha 0} - \Gamma^\alpha_{00}\Gamma^2_{\alpha 2} && \longrightarrow 0 - 0 + 0 - \Gamma^1_{00}\Gamma^2_{12} \\
&+\Gamma^3_{03,0} - \Gamma^3_{00,3} + \Gamma^\alpha_{03}\Gamma^3_{\alpha 0} - \Gamma^\alpha_{00}\Gamma^3_{\alpha 3} && \longrightarrow 0 - 0 + 0 - \Gamma^1_{00}\Gamma^3_{13}
\end{aligned}$$

Inserting the values we find,

$$\begin{aligned}
R_{00} &= -\Gamma^1_{00,1} + \Gamma^0_{01}\Gamma^1_{00} - \Gamma^1_{00}\Gamma^1_{11} - \Gamma^1_{00}\Gamma^2_{12} - \Gamma^1_{00}\Gamma^3_{13} \\
&= -\partial_r\frac{U'}{2V} + \frac{U'}{2U}\frac{U'}{2V} - \frac{U'}{2V}\frac{V'}{2V} - \frac{U'}{2V}\frac{1}{r} - \frac{U'}{2V}\frac{1}{r} \\
&= -\frac{U''}{2V} - \frac{U'}{2}\partial_r\frac{1}{V} + \frac{(U')^2}{4UV} - \frac{U'V'}{4V^2} - \frac{1}{r}\frac{U'}{V} \\
&= -\frac{U''}{2V} + \frac{U'}{2}\frac{V'}{V^2} + \frac{(U')^2}{4UV} - \frac{U'V'}{4V^2} - \frac{1}{r}\frac{U'}{V} \\
&= -\frac{U''}{2V} + \frac{U'}{4}\frac{V'}{V^2} + \frac{(U')^2}{4UV} - \frac{1}{r}\frac{U'}{V}
\end{aligned} \tag{6}$$

$$\begin{aligned}
R_{11} &= \Gamma^0_{10,1} - \Gamma^0_{11,0} + \Gamma^\alpha_{10}\Gamma^0_{\alpha 1} - \Gamma^\alpha_{11}\Gamma^0_{\alpha 0} && \longrightarrow \Gamma^0_{10,1} - 0 + \Gamma^0_{10}\Gamma^0_{01} - \Gamma^1_{11}\Gamma^0_{10} \\
&+\Gamma^1_{11,1} - \Gamma^1_{11,1} + \Gamma^\alpha_{11}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{11}\Gamma^1_{\alpha 1} && \longrightarrow \text{cancel pairwise} \\
&+\Gamma^2_{12,1} - \Gamma^2_{11,2} + \Gamma^\alpha_{12}\Gamma^2_{\alpha 1} - \Gamma^\alpha_{11}\Gamma^2_{\alpha 2} && \longrightarrow \Gamma^2_{12,1} - 0 + \Gamma^2_{12}\Gamma^2_{21} - \Gamma^1_{11}\Gamma^2_{12} \\
&+\Gamma^3_{13,1} - \Gamma^3_{11,3} + \Gamma^\alpha_{13}\Gamma^3_{\alpha 1} - \Gamma^\alpha_{11}\Gamma^3_{\alpha 3} && \longrightarrow \Gamma^3_{13,1} - 0 + \Gamma^3_{13}\Gamma^3_{31} - \Gamma^1_{11}\Gamma^3_{13}
\end{aligned}$$

Now collect terms and insert values,

$$\begin{aligned}
R_{11} &= \Gamma^0_{10,1} + \Gamma^0_{10}\Gamma^0_{01} - \Gamma^1_{11}\Gamma^0_{10} + \Gamma^2_{12,1} + \Gamma^2_{12}\Gamma^2_{21} - \Gamma^1_{11}\Gamma^2_{12} + \Gamma^3_{13,1} + \Gamma^3_{13}\Gamma^3_{31} - \Gamma^1_{11}\Gamma^3_{13} \\
&= \partial_r\frac{U'}{2U} + \frac{U'^2}{4U^2} - \frac{U'}{2U}\frac{V'}{2V} - \frac{1}{r^2} + \frac{1}{r^2} - \frac{V'}{2Vr} - \frac{1}{r^2} + \frac{1}{r^2} - \frac{V'}{2Vr} \\
&= \frac{U''}{2U} - \frac{U'^2}{2U^2} + \frac{U'^2}{4U^2} - \frac{U'}{2U}\frac{V'}{2V} - \frac{V'}{Vr} \\
&= \frac{U''}{2U} - \frac{U'^2}{4U^2} - \frac{U'}{2U}\frac{V'}{2V} - \frac{V'}{Vr}
\end{aligned}$$

$$\begin{aligned}
R_{22} &= \Gamma^0_{20,2} - \Gamma^0_{22,0} + \Gamma^\alpha_{20}\Gamma^0_{\alpha 2} - \Gamma^\alpha_{22}\Gamma^0_{\alpha 0} && \longrightarrow 0 - 0 + 0 - \Gamma^1_{22}\Gamma^0_{10} \\
&+\Gamma^1_{21,2} - \Gamma^1_{22,1} + \Gamma^\alpha_{21}\Gamma^1_{\alpha 2} - \Gamma^\alpha_{22}\Gamma^1_{\alpha 1} && \longrightarrow 0 - \Gamma^1_{22,1} + \Gamma^2_{21}\Gamma^1_{22} - \Gamma^1_{22}\Gamma^1_{11} \\
&+\Gamma^2_{22,2} - \Gamma^2_{22,2} + \Gamma^\alpha_{22}\Gamma^2_{\alpha 2} - \Gamma^\alpha_{22}\Gamma^2_{\alpha 2} && \longrightarrow \text{cancel pairwise} \\
&+\Gamma^3_{23,2} - \Gamma^3_{22,3} + \Gamma^\alpha_{23}\Gamma^3_{\alpha 2} - \Gamma^\alpha_{22}\Gamma^3_{\alpha 3} && \longrightarrow \Gamma^3_{23,2} - 0 + \Gamma^3_{23}\Gamma^3_{32} - \Gamma^1_{22}\Gamma^3_{13}
\end{aligned}$$

Collecting terms and inserting values we get,

$$\begin{aligned}
R_{22} &= -\Gamma^1_{22}\Gamma^0_{10} - \Gamma^1_{22,1} + \Gamma^2_{21}\Gamma^1_{22} - \Gamma^1_{22}\Gamma^1_{11} + \Gamma^3_{23,2} + \Gamma^3_{23}\Gamma^3_{32} - \Gamma^1_{22}\Gamma^3_{13} \\
&= \frac{r}{V}\frac{U'}{2U} + (\frac{1}{V} - \frac{rV'}{V^2}) - \frac{1}{V} + \frac{rV'}{2V^2} + \partial_\theta \cot\theta + \cot^2\theta + \frac{1}{V} \\
&= \frac{rU'}{2UV} + \frac{1}{V} - \frac{rV'}{2V^2} + (-1 - \cot^2\theta) + \cot^2\theta \\
&= \frac{rU'}{2UV} + \frac{1}{V} - \frac{rV'}{2V^2} - 1
\end{aligned}$$

$$\begin{aligned}
R_{33} = \;& \Gamma^0_{30,3} - \Gamma^0_{33,0} + \Gamma^\alpha_{30}\Gamma^0_{\alpha 3} - \Gamma^\alpha_{33}\Gamma^0_{\alpha 0} && \longrightarrow 0 - 0 + 0 - \Gamma^1_{33}\Gamma^0_{10} \\
& +\Gamma^1_{31,3} - \Gamma^1_{33,1} + \Gamma^\alpha_{31}\Gamma^1_{\alpha 3} - \Gamma^\alpha_{33}\Gamma^1_{\alpha 1} && \longrightarrow 0 - \Gamma^1_{33,1} + \Gamma^3_{31}\Gamma^1_{33} - \Gamma^1_{33}\Gamma^1_{11} \\
& +\Gamma^2_{32,3} - \Gamma^2_{33,2} + \Gamma^\alpha_{32}\Gamma^2_{\alpha 3} - \Gamma^\alpha_{33}\Gamma^2_{\alpha 2} && \longrightarrow 0 - \Gamma^2_{33,2} + \Gamma^3_{32}\Gamma^2_{33} - \Gamma^1_{33}\Gamma^2_{12} \\
& +\Gamma^3_{33,3} - \Gamma^3_{33,3} + \Gamma^\alpha_{33}\Gamma^3_{\alpha 3} - \Gamma^\alpha_{33}\Gamma^3_{\alpha 3} && \longrightarrow \text{cancel pairwise}
\end{aligned}$$

Collecting terms and inserting values we get,

$$\begin{aligned}
R_{33} &= -\Gamma^1_{33}\Gamma^0_{10} - \Gamma^1_{33,1} + \Gamma^3_{31}\Gamma^1_{33} - \Gamma^1_{33}\Gamma^1_{11} - \Gamma^2_{33,2} + \Gamma^3_{32}\Gamma^2_{33} - \Gamma^1_{33}\Gamma^2_{12} \\
&= \frac{rU'}{2UV}\sin^2\theta + (\frac{\sin^2\theta}{V} - \frac{r\sin^2\theta V'}{V^2}) - \frac{\sin^2\theta}{V} + \frac{rV'}{2V^2}\sin^2\theta + \partial_\theta(\cos\theta\sin\theta) - \cos\theta\sin\theta\cot\theta + \frac{\sin^2\theta}{V} \\
&= \frac{rU'}{2UV}\sin^2\theta + \frac{\sin^2\theta}{V} - \frac{rV'}{2V^2}\sin^2\theta + (-\sin^2\theta + \cos^2\theta) - \cos^2\theta \\
&= \frac{rU'}{2UV}\sin^2\theta + \frac{\sin^2\theta}{V} - \frac{rV'}{2V^2}\sin^2\theta - \sin^2\theta \\
&= \left(\frac{rU'}{2UV} + \frac{1}{V} - \frac{rV'}{2V^2} - 1\right)\sin^2\theta = \sin^2\theta R_{22}
\end{aligned}$$

The Ricci scalar

The Ricci scalar is obtained from the Ricci tensor,

$$\begin{aligned}
R &= R^\mu_\mu = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}\sin^2\theta R_{22} \\
&= \frac{1}{U}R_{00} - \frac{1}{V}R_{11} - \frac{1}{r^2}R_{22} - (\frac{1}{r^2\sin^2\theta})\sin^2\theta R_{22} \\
&= \frac{1}{U}R_{00} - \frac{1}{V}R_{11} - \frac{2}{r^2}R_{22} \\
&= -\frac{U''}{2UV} + \frac{U'}{4}\frac{V'}{UV^2} + \frac{(U')^2}{4U^2V} - \frac{1}{r}\frac{U'}{UV} \\
&\quad -\frac{U''}{2UV} + \frac{U'^2}{4U^2V} + \frac{U'}{2U}\frac{V'}{2V^2} + \frac{V'}{V^2 r} \\
&\quad -\frac{U'}{rUV} - \frac{2}{r^2V} + \frac{V'}{rV^2} + \frac{2}{r^2} \\
&= -\frac{U''}{UV} + \frac{U'}{2}\frac{V'}{UV^2} + \frac{(U')^2}{2U^2V} - \frac{2}{r}\frac{U'}{UV} + \frac{2}{r}\frac{V'}{V^2} + \frac{2}{r^2}(1 - \frac{1}{V})
\end{aligned}$$

2 The Einstein equation for Schwarzschild spacetime

Since the Schwarzschild solution concerns with the exterior spacetime of a spherically symmetric body, Einstein's equation (3) takes a simpler form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{7}$$

This leads to four equations that must be satisfied.

$$
\begin{aligned}
R_{00} - \frac{1}{2} g_{00} R &= 0 = R_{00} - \frac{U}{2} R \\
0 &= -\frac{U''}{2V} + \frac{U'}{4}\frac{V'}{V^2} + \frac{(U')^2}{4UV} - \frac{1}{r}\frac{U'}{V} + \frac{U''}{2V} - \frac{U'}{4}\frac{V'}{V^2} - \frac{(U')^2}{4UV} + \frac{1}{r}\frac{U'}{V} + \frac{1}{r}\frac{UV'}{V^2} + \frac{U}{r^2}(1 - \frac{1}{V}) \\
0 &= \frac{1}{r}\frac{V'}{V^2} + \frac{1}{r^2}(1 - \frac{1}{V}) \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
R_{11} - \frac{1}{2} g_{11} R &= 0 = R_{11} + \frac{V}{2} R \\
0 &= \frac{U''}{2U} - \frac{U'^2}{4U^2} - \frac{U'V'}{4UV} - \frac{V'}{Vr} - \frac{U''}{2U} + \frac{U'}{4}\frac{V'}{UV} + \frac{(U')^2}{4U^2} - \frac{1}{r}\frac{U'}{U} + \frac{1}{r}\frac{V'}{V} + \frac{V}{r^2}(1 - \frac{1}{V}) \\
0 &= -\frac{1}{r}\frac{U'}{U} + \frac{V}{r^2}(1 - \frac{1}{V}) \\
0 &= -\frac{U'}{rUV} + \frac{1}{r^2}(1 - \frac{1}{V}) \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
R_{22} - \frac{1}{2} g_{22} R &= 0 = R_{22} + \frac{r^2}{2} R \\
0 &= \frac{rU'}{2UV} + \frac{1}{V} - \frac{rV'}{2V^2} - 1 - \frac{r^2 U''}{2UV} + \frac{r^2 U'}{4}\frac{V'}{UV^2} + \frac{r^2 (U')^2}{4U^2 V} - r\frac{U'}{UV} + r\frac{V'}{V^2} + (1 - \frac{1}{V}) \\
0 &= -\frac{rU'}{2UV} + \frac{rV'}{2V^2} - \frac{r^2 U''}{2UV} + \frac{r^2 U'}{4}\frac{V'}{UV^2} + \frac{r^2 (U')^2}{4U^2 V} \\
0 &= -\frac{U'}{U} + \frac{V'}{V} - \frac{rU''}{U} + \frac{rU'}{2}\frac{V'}{UV} + \frac{r(U')^2}{2U^2}
\end{aligned}
$$

Note that the last equation is not independent of the previous,

$$
R_{33} - \frac{1}{2} g_{33} R = 0 = \sin^2\theta R_{22} + \frac{r^2 \sin^2\theta}{2} R = R_{22} + \frac{r^2}{2} R \qquad (10)
$$

2.1 Solving the Einstein equation

We are nearly there. Notice that the first equation for G_{00}, (8), is only a function of V. Thus we can solve this differential equation to find V and then use that in the other equations to find U. Re-expressing that equation we get,

$$
\begin{aligned}
0 &= \frac{V'}{V} + \frac{1}{r}(V - 1) \\
0 &= \frac{V'}{V(V-1)} + \frac{1}{r} \\
-\frac{dr}{r} &= \frac{dV}{V(V-1)} \qquad (11)
\end{aligned}
$$

Integrating this and using the formula $\int \frac{dx}{ax+bx^2} = -\frac{1}{a} \ln \frac{a+bx}{x}$ we get,

$$
\begin{aligned}
-\ln r + C' = \ln \frac{1}{r} + C' &= \ln \frac{(V-1)}{V} \\
\frac{C}{r} &= \frac{V-1}{V} \\
V - 1 &= \frac{C}{r} V \qquad \longrightarrow \qquad V(1 - \frac{C}{r}) = 1 \\
V &= \frac{1}{1 - \frac{C}{r}}
\end{aligned}
$$

This is half of our solution.

To find U we insert this solution into (9) to get,

$$\begin{aligned}0 &= \frac{U'}{U}(1-\frac{C}{r}) - \frac{1}{r}(1-(1-\frac{C}{r}))\\ 0 &= \frac{U'}{U}(1-\frac{C}{r}) - \frac{C}{r^2}\\ \frac{U'}{U} &= \frac{C}{r^2(1-\frac{C}{r})} = \frac{C}{r^2-Cr}\\ \frac{dU}{U} &= \frac{Cdr}{r^2-Cr}\end{aligned}$$

Now we integrate both sides of this equation (using the same formula we used before) to get,

$$\int \frac{dU}{U} = \ln U = C\int \frac{dr}{r^2-Cr} = \frac{C}{C}\ln\frac{r-C}{r} = \ln(1-\frac{C}{r})$$

Exponentiating both sides we obtain are other answer.

$$U = (1-\frac{C}{r})$$

Thus our Schwarzschild metric is,

$$ds^2 = (1-\frac{C}{r})c^2dt^2 - \frac{dr^2}{(1-\frac{C}{r})} - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 \tag{12}$$

First we notice that this metric satisfies our properties spelled out at the beginning.

- It is spherically symmetric.
- It is asymptotically flat (when $r \longrightarrow \infty$ it is our flat spacetime metric).

One more requirement will help us pin down the remaining undetermined parameter C. We expect that in the limit that $M \longrightarrow 0$ we should again obtain the flat metric. We see that this would result when $C = 0$ so we hypothesize that it is proportional to mass (as there are no other free parameters this must be the case). To get the exact form we need to compare the results for geodesics for this metric for a very small test mass when the source mass, M, is weak. Or comparing Einstein's equation in the low mass limit and noting we must regain Newton's law will also work. The result is that $C = \frac{2GM}{c^2}$ and yields the full Schwarzschild metric.

$$ds^2 = (1-\frac{2GM}{c^2r})c^2dt^2 - \frac{dr^2}{(1-\frac{2GM}{c^2r})} - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 \tag{13}$$

Some loose ends

Birkhoff's theorem

Recall at the outset that we stated that the coefficients U, V were not functions of time. Well, you may not believe that. To convince you, we only need to look at one other term in the Ricci tensor but no longer assume time-independence. So return to examine the term $R_{0r} = R_{01}$. But first we have to derive the additional Christoffel symbols that no longer need to vanish due to time independence. The additional terms are,

$$\Gamma^0_{00} = -\frac{\dot{U}}{2U}, \qquad \Gamma^0_{11} = \frac{\dot{V}}{2U}, \qquad \Gamma^1_{01} = \Gamma^1_{10} = \frac{\dot{V}}{2V}$$

Now returning to this term of the Ricci tensor,

$$\begin{aligned}R_{01} = {} & \Gamma^0_{00,1} - \Gamma^0_{01,0} + \Gamma^\alpha_{00}\Gamma^0_{\alpha 1} - \Gamma^\alpha_{01}\Gamma^0_{\alpha 0} && \longrightarrow \partial_r\Gamma^0_{00} - 0 + \Gamma^\alpha_{00}\Gamma^0_{\alpha 1} - \Gamma^\alpha_{01}\Gamma^0_{\alpha 0}\\ & +\Gamma^1_{01,1} - \Gamma^1_{01,1} + \Gamma^\alpha_{01}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{0i}\Gamma^1_{\alpha 1} && \text{cancel pairwise}\\ & +\Gamma^2_{02,1} - \Gamma^2_{01,2} + \Gamma^\alpha_{02}\Gamma^2_{\alpha 1} - \Gamma^\alpha_{0i}\Gamma^2_{\alpha 2} && \longrightarrow 0 - 0 + 0 - 0\\ & +\Gamma^3_{03,1} - \Gamma^3_{01,3} + \Gamma^\alpha_{03}\Gamma^3_{\alpha 1} - \Gamma^\alpha_{0i}\Gamma^3_{\alpha 3} && \longrightarrow 0 - 0 + 0 - 0\end{aligned}$$

We are left to examine,

$$\begin{aligned}
R_{01} &= \partial_r\Gamma^0_{00} + \Gamma^\alpha_{00}\Gamma^0_{\alpha 1} - \Gamma^\alpha_{01}\Gamma^0_{\alpha 0} = \partial_r\Gamma^0_{00} + \Gamma^0_{00}\Gamma^0_{01} - \Gamma^0_{01}\Gamma^0_{00} + \Gamma^1_{00}\Gamma^0_{11} - \Gamma^1_{01}\Gamma^0_{10} \\
&= \partial_r\Gamma^0_{00} + \Gamma^1_{00}\Gamma^0_{11} - \Gamma^1_{01}\Gamma^0_{10} \\
&= -\partial_r\frac{\dot{U}}{2U} + \left(\frac{U'}{2V}\frac{\dot{V}}{2U}\right) - \left(\frac{\dot{V}}{2V}\frac{U'}{2U}\right) = -\partial_r\frac{\dot{U}}{2U} \\
&= -\frac{1}{2U}\frac{\partial}{\partial r}\frac{\partial}{\partial t}U + \frac{1}{2U^2}\frac{\partial U}{\partial t}\frac{\partial U}{\partial r} \qquad (14)
\end{aligned}$$

Note that, from Einstein's equation, this term must vanish: $R_{01} - \frac{1}{2}g_{01}R = 0 = R_{01}$, or,

$$\begin{aligned}
0 &= -\frac{1}{2U}\frac{\partial}{\partial r}\frac{\partial}{\partial t}U + \frac{1}{2U^2}\frac{\partial U}{\partial t}\frac{\partial U}{\partial r} \\
0 &= \frac{\partial}{\partial r}\frac{\partial}{\partial t}U - \frac{1}{U}\frac{\partial U}{\partial t}\frac{\partial U}{\partial r} = \frac{\partial}{\partial r}\frac{\partial}{\partial t}U - \frac{1}{U}\frac{\partial}{\partial r}\left(U\frac{\partial U}{\partial t}\right) + \frac{\partial}{\partial r}\frac{\partial}{\partial t}U
\end{aligned} \qquad (15)$$

3 Other Coordinates in Schwarzschild spacetime

Eddinginton-Finkelstein inward and outward coordinates

To analytically continue the Schwarzschild coordinates through the horizon and down to the singularity we define a new radial coordinate (with $R_s = \frac{2GM}{c^2}$),

$$\begin{aligned}
r^* &\equiv \int \frac{dr}{1 - \frac{R_s}{r}}, \qquad \text{define, } x = r - R_s, \\
&= \int \frac{x + R_s}{x}dx = \int 1 + \frac{R_s}{x}dx = x + R_s \ln x, \\
&= r - R_s + \ln(r - R_s) \qquad (16)
\end{aligned}$$

We will ignore the constant, R_s and stick with $r^* = r + R_s \ln(r - R_s)$.

Now define the advanced null coordinate,

$$v = ct - r^*$$

and retarded null coordinate,

$$w = ct + r^*$$